1. Let f be continuous and differentiable, and suppose f has 2 roots. Prove that f' must also have atleast one root. What can we say about the location of the root of f'.

2. Let f, g be continuous on $[a, b]$ and differentiable on (a, b). Suppose $f(a) = f(b)$ and $g(a) = g(b)$. Prove that there is a $c \in (a, b)$ such that $f'(c) = g'(c)$.

3. Let f be twice differentiable on $[0, 2]$, show that if $f(0) = 0, f(1) = 2, f(2) = 4$, then there is an $x_0 \in (0, 2)$ such that $f''(x_0) = 0$.

4. Let f be continuous on $[-a, a]$ and differentiable on $(-a, a)$. Suppose that $f'(x) \leq 1$ for $x \in (-a, a)$ and $f(a) = a, f(-a) = -a$. Prove that $f(x) = x$.
   Hint: Let $g(x) = f(x) - x$. What can you say about $g'(x)$?

5. **(Hard)** Suppose f is continuous on $[1, \infty)$, and differentiable on $(1, \infty)$. Determine if the following are true or false. If true provide a proof, if false provide a counter-example.

   (a) If $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} f'(x)$ exists, then $\lim_{x\to\infty} f'(x) = 0$. **Hint:** Apply MVT to $[n, n+1]$ for $n = 1, 2, 3, \ldots$ to get a sequence c_n. Then what can you say about $\lim_{n\to\infty} f'(c_n)$?

   (b) If $\lim_{x\to\infty} f(x) = 0$ then $\lim_{x\to\infty} f'(x) = 0$.

Solution.

1. Let's suppose a, b are the two roots of f with $a < b$. So $f(a) = f(b) = 0$. So by Rolle's theorem, there is a $c \in (a, b)$ such that $f'(c) = 0$.

2. Let $h(x) = f(x) - g(x)$. h is continuous on $[a, b]$ and differentiable on (a, b). Also since $h(a) = h(b) = 0$, by Rolle's theorem there is a $c \in (a, b)$ such that $h'(c) = 0$, i.e.,
$$f'(c) = g'(c).$$

3. By mean value theorem we have a $x_1 \in (0, 1)$ such that
$$f'(x_1) = \frac{f(1) - f(0)}{1 - 0} = 2.$$
Again by mean value theorem there is a $x_2 \in (1, 2)$ such that
$$f'(x_2) = \frac{f(2) - f(1)}{2 - 1} = 2.$$
Since f is differentiable on $[x_1, x_2]$ and twice differentiable on (x_1, x_2), and $f'(x_1) = f'(x_2)$, by applying Rolle's theorem to f' we get there is a $x_0 \in (x_1, x_2)$ such that $f''(x_0) = 0$.

4. Let $g(x) = f(x) - x$. g is continuous on $[-a, a]$ and differentiable on $(-a, a)$ since f is. We also have $g(a) = g(-a) = 0$ and $g'(x) = f'(x) - 1 \leq 0$, since $f'(x) \leq 1$. We are done if we can show that $g(x) = 0$ for $x \in (-a, a)$. By applying MVT to $[-a, x]$, we have there is a $x_1 \in (-a, x)$ such that,
$$\frac{g(x) - g(-a)}{x - (-a)} = \frac{g(x)}{x + a} = g'(x_1) \leq 0.$$
So $g(x) \leq 0$. Now by applying MVT to $[x, a]$, we have there is a $x_2 \in (x, a)$ such that,
$$\frac{g(a) - g(x)}{a - x} = \frac{-g(x)}{a - x} = g'(x_2) \leq 0.$$
So $-g(x) \leq 0$ or $g(x) \geq 0$. So putting the two inequalities together we get $g(x) = 0$ for all $x \in [-a, a]$. Therefore $f(x) = x$ on $[-a, a]$.

5. (a) **TRUE**
   Let's apply MVT to $[n, n+1]$ for $n = 1, 2, 3 \ldots$. We have there is a $c_n \in (n, n+1)$ such that
$$f'(c_n) = \frac{f(n+1) - f(n)}{n + 1 - n} = f(n+1) - f(n)$$

So we have an increasing sequence of c_n and since $n < c_n$ we have c_n are going to infinity as n goes to infinity. Since $\lim_{x\to\infty} f'(x) = L$ for some L, and c_n is a sequence going to infinity, we have

$$\begin{aligned} L &= \lim_{x\to\infty} f'(x) \\ &= \lim_{n\to\infty} f'(c_n) \qquad \text{why?} \\ &= \lim_{n\to\infty} f(n+1) - f(n) \\ &= \lim_{n\to\infty} f(n+1) - \lim_{n\to\infty} f(n) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

Where the second last line is true because $\lim_{x\to\infty} f(x) = 0$. Thus

$$\lim_{x\to\infty} f'(x) = 0.$$

(b) **FALSE**
It is possible that even though f is going to 0, that f' is fluctuating very rapidly as x gets really large. For example look at the function

$$f(x) = \frac{\sin(x^2)}{x}.$$

We have $\lim_{x\to\infty} f(x) = 0$ since

$$|f(x)| = \left|\frac{\sin(x^2)}{x}\right| \leq \frac{1}{|x|} \xrightarrow[x\to\infty]{} 0.$$

Now let's compute $f'(x)$. By quotient rule we have,

$$f'(x) = \frac{\cos(x^2)2x \cdot x - \sin(x^2) \cdot 1}{x^2} = 2\cos(x^2) - \frac{\sin(x^2)}{x}$$

The limit of $f'(x)$ as x goes to infinity does not exist (why?).